

ON PARTIAL SUMS OF NORMALIZED q -BESSEL FUNCTIONS

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ABSTRACT. In the present investigation our main aim is to give lower bounds for the ratio of some normalized q -Bessel functions and their sequences of partial sums. Especially, we consider Jackson's second and third q -Bessel functions and we apply one normalization for each of them.

1. INTRODUCTION

Let \mathcal{A} denote the class of functions of the following form:

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We denote by \mathcal{S} the class of all functions in \mathcal{A} which are univalent in \mathcal{U} .

The Jackson's second and third q -Bessel functions are defined by (see [4])

$$(1.2) \quad J_{\nu}^{(2)}(z; q) = \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} \sum_{n \geq 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{(q; q)_n (q^{\nu+1}; q)_n} q^{n(n+\nu)}$$

and

$$(1.3) \quad J_{\nu}^{(3)}(z; q) = \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} \sum_{n \geq 0} \frac{(-1)^n z^{2n+\nu}}{(q; q)_n (q^{\nu+1}; q)_n} q^{\frac{1}{2}n(n+1)},$$

where $z \in \mathbb{C}, \nu > -1, q \in (0, 1)$ and

$$(a; q)_0 = 1, (a; q)_n = \prod_{k=1}^n (1 - aq^{k-1}), (a, q)_{\infty} = \prod_{k \geq 1} (1 - aq^{k-1}).$$

Here we would like to say that Jackson's third q -Bessel function is also known as Hahn-Exton q -Bessel function.

Recently, the some geometric properties like univalence, starlikeness and convexity of the some special functions were investigated by many authors. Especially, in [1, 5, 6, 8] authors have studied on the starlikeness and convexity of the some normalized q -Bessel functions. In addition, the some lower bounds for the ratio of some special functions and their sequences of partial sums were given in [3, 7, 10, 11]. Moreover, results related with partial sums of analytic functions can be found in [2, 9, 12, 13, 14] etc.

Motivated by the previous works on analytic and some special functions, in this paper our aim is to present some lower bounds for the ratio of normalized q -Bessel functions to their sequences of partial sums.

Due to the functions defined by (1.2) and (1.3) do not belong to the class \mathcal{A} , we consider following normalized forms of the q -Bessel functions:

$$(1.4) \quad h_\nu^{(2)}(z; q) = 2^\nu c_\nu(q) z^{1-\frac{\nu}{2}} J_\nu^{(2)}(\sqrt{z}; q) = \sum_{n \geq 0} K_n z^{n+1}$$

and

$$(1.5) \quad h_\nu^{(3)}(z; q) = c_\nu(q) z^{1-\frac{\nu}{2}} J_\nu^{(3)}(\sqrt{z}; q) = \sum_{n \geq 0} T_n z^{n+1},$$

where $K_n = \frac{(-1)^n q^{n(n+\nu)}}{4^n (q; q)_n (q^{\nu+1}; q)_n}$, $T_n = \frac{(-1)^n q^{\frac{1}{2}n(n+1)}}{(q; q)_n (q^{\nu+1}; q)_n}$ and $c_\nu(q) = (q; q)_\infty / (q^{\nu+1}; q)_\infty$. As a result of the above normalizations, all of the above functions belong to the class \mathcal{A} .

2. MAIN RESULTS

The following lemmas will be required in order to derive our main results.

Lemma 1. *Let $q \in (0, 1)$, $\nu > -1$ and $4(1-q)(1-q^\nu) > q^\nu$. Then the function $h_\nu^{(2)}(z; q)$ satisfies the next two inequalities for $z \in \mathcal{U}$:*

$$(2.1) \quad |h_\nu^{(2)}(z; q)| \leq \frac{4(1-q)(1-q^\nu)}{4(1-q)(1-q^\nu) - q^\nu},$$

$$(2.2) \quad |(h_\nu^{(2)}(z; q))'| \leq \left(\frac{4(1-q)(1-q^\nu)}{4(1-q)(1-q^\nu) - q^\nu} \right)^2.$$

Proof. It can be easily shown that the inequalities

$$q^{n(n+\nu)} \leq q^{n\nu}, (1-q)^n \leq (q; q)_n \text{ and } (1-q^\nu)^n \leq (q^{\nu+1}; q)_n$$

are valid for $q \in (0, 1)$ and $\nu > -1$. Making use the above inequalities and well-known triangle inequality, for $z \in \mathcal{U}$, we get

$$\begin{aligned} |h_\nu^{(2)}(z; q)| &= \left| z + \sum_{n \geq 1} \frac{(-1)^n q^{n(n+\nu)}}{4^n (q; q)_n (q^{\nu+1}; q)_n} z^{n+1} \right| \\ &\leq 1 + \sum_{n \geq 1} \frac{q^{n(n+\nu)}}{4^n (q; q)_n (q^{\nu+1}; q)_n} \\ &\leq 1 + \sum_{n \geq 1} \left(\frac{q^\nu}{4(1-q)(1-q^\nu)} \right)^n \\ &= 1 + \frac{q^\nu}{4(1-q)(1-q^\nu)} \sum_{n \geq 1} \left(\frac{q^\nu}{4(1-q)(1-q^\nu)} \right)^{n-1} \\ &= \frac{4(1-q)(1-q^\nu)}{4(1-q)(1-q^\nu) - q^\nu} \end{aligned}$$

and

$$\begin{aligned}
\left| (h_\nu^{(2)}(z; q))' \right| &= \left| 1 + \sum_{n \geq 1} \frac{(-1)^n (n+1) q^{n(n+\nu)}}{4^n (q; q)_n (q^{\nu+1}; q)_n} z^n \right| \\
&\leq 1 + \sum_{n \geq 1} \frac{(n+1) q^{n(n+\nu)}}{4^n (q; q)_n (q^{\nu+1}; q)_n} \\
&\leq 1 + \sum_{n \geq 1} (n+1) \left(\frac{q^\nu}{4(1-q)(1-q^\nu)} \right)^n \\
&= \left(\frac{4(1-q)(1-q^\nu)}{4(1-q)(1-q^\nu) - q^\nu} \right)^2.
\end{aligned}$$

Thus, the inequalities (2.1) and (2.2) are proved. \square

Lemma 2. *Let $q \in (0, 1)$, $\nu > -1$ and $(1-q)(1-q^\nu) > \sqrt{q}$. Then the function $h_\nu^{(3)}(z; q)$ satisfies the inequalities*

$$(2.3) \quad \left| h_\nu^{(3)}(z; q) \right| \leq \frac{(1-q)(1-q^\nu)}{(1-q)(1-q^\nu) - \sqrt{q}},$$

and

$$(2.4) \quad \left| (h_\nu^{(3)}(z; q))' \right| \leq \left(\frac{(1-q)(1-q^\nu)}{(1-q)(1-q^\nu) - \sqrt{q}} \right)^2$$

for $z \in \mathcal{U}$.

Proof. It is known that the inequalities

$$q^{\frac{1}{2}n(n+1)} \leq q^{\frac{1}{2}n}, (1-q)^n \leq (q; q)_n \text{ and } (1-q^\nu)^n \leq (q^{\nu+1}; q)_n$$

are valid for $q \in (0, 1)$ and $\nu > -1$. Now, using the well-known triangle inequality for $z \in \mathcal{U}$, we have

$$\begin{aligned}
\left| h_\nu^{(3)}(z; q) \right| &= \left| z + \sum_{n \geq 1} \frac{(-1)^n q^{\frac{1}{2}n(n+1)}}{(q; q)_n (q^{\nu+1}; q)_n} z^{n+1} \right| \\
&\leq 1 + \sum_{n \geq 1} \frac{q^{\frac{1}{2}n}}{(1-q)^n (1-q^\nu)^n} \\
&\leq 1 + \frac{\sqrt{q}}{(1-q)(1-q^\nu)} \sum_{n \geq 1} \left(\frac{\sqrt{q}}{(1-q)(1-q^\nu)} \right)^{n-1} \\
&= \frac{(1-q)(1-q^\nu)}{(1-q)(1-q^\nu) - \sqrt{q}}
\end{aligned}$$

and

$$\begin{aligned}
\left| (h_\nu^{(3)}(z; q))' \right| &= \left| 1 + \sum_{n \geq 1} \frac{(-1)^n (n+1) q^{\frac{1}{2}n(n+1)}}{(q; q)_n (q^{\nu+1}; q)_n} z^n \right| \\
&\leq 1 + \sum_{n \geq 1} (n+1) \frac{q^{\frac{1}{2}n}}{(1-q)^n (1-q^\nu)^n} \\
&\leq 1 + \frac{\sqrt{q}}{(1-q)(1-q^\nu)} \sum_{n \geq 1} (n+1) \left(\frac{\sqrt{q}}{(1-q)(1-q^\nu)} \right)^{n-1} \\
&= \left(\frac{(1-q)(1-q^\nu)}{(1-q)(1-q^\nu) - \sqrt{q}} \right)^2.
\end{aligned}$$

So, the inequalities (2.3) and (2.4) are proved. \square

Let $w(z)$ denote an analytic function in \mathcal{U} . In the proof of our main results, the following well-known result will be used frequently:

$$\Re \left\{ \frac{1+w(z)}{1-w(z)} \right\} > 0, \text{ if and only if } |w(z)| < 1, z \in \mathcal{U}.$$

Theorem 1. *Let $\nu > -1, q \in (0, 1)$, the function $h_\nu^{(2)} : \mathcal{U} \rightarrow \mathbb{C}$ be defined by (1.4) and its sequences of partial sums by $(h_\nu^{(2)})_m(z; q) = z + \sum_{n=1}^m K_n z^{n+1}$. If the inequality $2(1-q)(1-q^\nu) \geq q^\nu$, then the following inequalities hold true for $z \in \mathcal{U}$:*

$$(2.5) \quad \Re \left\{ \frac{h_\nu^{(2)}(z; q)}{(h_\nu^{(2)})_m(z; q)} \right\} \geq \frac{4(1-q)(1-q^\nu) - 2q^\nu}{4(1-q)(1-q^\nu) - q^\nu},$$

$$(2.6) \quad \Re \left\{ \frac{(h_\nu^{(2)})_m(z; q)}{h_\nu^{(2)}(z; q)} \right\} \geq \frac{4(1-q)(1-q^\nu) - q^\nu}{4(1-q)(1-q^\nu)}.$$

Proof. From the inequality (2.1) we have that

$$(2.7) \quad 1 + \sum_{n \geq 1} |K_n| \leq \frac{4(1-q)(1-q^\nu)}{4(1-q)(1-q^\nu) - q^\nu}.$$

The inequality (2.7) is equivalent to

$$(2.8) \quad \frac{4(1-q)(1-q^\nu) - q^\nu}{q^\nu} \sum_{n \geq 1} |K_n| \leq 1.$$

In order to prove the inequality (2.5), we consider the function $w(z)$ defined by

$$\frac{1+w(z)}{1-w(z)} = \frac{4(1-q)(1-q^\nu) - q^\nu}{q^\nu} \left\{ \frac{h_\nu^{(2)}(z; q)}{(h_\nu^{(2)})_m(z; q)} - \frac{4(1-q)(1-q^\nu) - 2q^\nu}{4(1-q)(1-q^\nu) - q^\nu} \right\}$$

which is equivalent to

$$(2.9) \quad \frac{1+w(z)}{1-w(z)} = \frac{1 + \sum_{n=1}^m K_n z^n + \frac{4(1-q)(1-q^\nu) - q^\nu}{q^\nu} \sum_{n=m+1}^{\infty} K_n z^n}{1 + \sum_{n=1}^m K_n z^n}.$$

By using the equality (2.9) we get

$$w(z) = \frac{\frac{4(1-q)(1-q^\nu)-q^\nu}{q^\nu} \sum_{n=m+1}^{\infty} K_n z^n}{2 + 2 \sum_{n=1}^m K_n z^n + \frac{4(1-q)(1-q^\nu)-q^\nu}{q^\nu} \sum_{n=m+1}^{\infty} K_n z^n}$$

and

$$|w(z)| \leq \frac{\frac{4(1-q)(1-q^\nu)-q^\nu}{q^\nu} \sum_{n=m+1}^{\infty} |K_n|}{2 - 2 \sum_{n=1}^m |K_n| - \frac{4(1-q)(1-q^\nu)-q^\nu}{q^\nu} \sum_{n=m+1}^{\infty} |K_n|}.$$

The inequality

$$(2.10) \quad \sum_{n=1}^m |K_n| + \frac{4(1-q)(1-q^\nu)-q^\nu}{q^\nu} \sum_{n=m+1}^{\infty} |K_n| \leq 1$$

implies that $|w(z)| \leq 1$. It suffices to show that the left hand side of (2.10) is bounded above by

$$\frac{4(1-q)(1-q^\nu)-q^\nu}{q^\nu} \sum_{n \geq 1} |K_n|,$$

which is equivalent to

$$\frac{4(1-q)(1-q^\nu)-2q^\nu}{q^\nu} \sum_{n \geq 1} |K_n| \geq 0.$$

The last inequality holds true for $2(1-q)(1-q^\nu) \geq q^\nu$.

In order to prove the result (2.6) we use the same method. Now, consider the function $p(z)$ given by

$$\frac{1+p(z)}{1-p(z)} = \left(1 + \frac{4(1-q)(1-q^\nu)-q^\nu}{q^\nu}\right) \left\{ \frac{(h_\nu^{(2)})_m(z; q)}{h_\nu^{(2)}(z; q)} - \frac{4(1-q)(1-q^\nu)-q^\nu}{4(1-q)(1-q^\nu)} \right\}.$$

Then from the last equality we get

$$p(z) = \frac{-\frac{4(1-q)(1-q^\nu)}{q^\nu} \sum_{n=m+1}^{\infty} K_n z^n}{2 + 2 \sum_{n=1}^m K_n z^n - \frac{4(1-q)(1-q^\nu)}{q^\nu} \sum_{n=m+1}^{\infty} K_n z^n}$$

and

$$|p(z)| \leq \frac{\frac{4(1-q)(1-q^\nu)}{q^\nu} \sum_{n=m+1}^{\infty} |K_n|}{2 - 2 \sum_{n=1}^m |K_n| - \frac{4(1-q)(1-q^\nu)}{q^\nu} \sum_{n=m+1}^{\infty} |K_n|}.$$

The inequality

$$(2.11) \quad \sum_{n=1}^m |K_n| + \frac{4(1-q)(1-q^\nu)}{q^\nu} \sum_{n=m+1}^{\infty} |K_n| \leq 1$$

implies that $|p(z)| \leq 1$. Since the left hand side of (2.11) is bounded above by

$$\frac{4(1-q)(1-q^\nu)-q^\nu}{q^\nu} \sum_{n=1}^m |K_n| \geq 0$$

the proof is completed. \square

Theorem 2. Let $\nu > -1, q \in (0, 1)$, the function $h_\nu^{(2)} : \mathcal{U} \rightarrow \mathbb{C}$ be defined by (1.4) and its sequences of partial sums by $(h_\nu^{(2)})_m(z; q) = z + \sum_{n=1}^m K_n z^{n+1}$. If the inequality $(1-q)(1-q^\nu) \geq q^\nu$ is valid, then the following inequalities hold true for $z \in \mathcal{U}$:

$$(2.12) \quad \Re \left\{ \frac{\left(h_\nu^{(2)}(z; q) \right)'}{\left((h_\nu^{(2)})_m(z; q) \right)'} \right\} \geq \frac{16(1-q)(1-q^\nu) \left((1-q)(1-q^\nu) - q^\nu \right) + 2q^{2\nu}}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}},$$

$$(2.13) \quad \Re \left\{ \frac{\left((h_\nu^{(2)})_m(z; q) \right)'}{\left(h_\nu^{(2)}(z; q) \right)'} \right\} \geq \frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}}.$$

Proof. From the inequality (2.2) we have that

$$(2.14) \quad 1 + \sum_{n \geq 1} (n+1) |K_n| \leq \left(\frac{4(1-q)(1-q^\nu)}{4(1-q)(1-q^\nu) - q^\nu} \right)^2.$$

The inequality (2.14) is equivalent to

$$(2.15) \quad \frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n \geq 1} (n+1) |K_n| \leq 1.$$

In order to prove the inequality (2.12), we consider the function $h(z)$ defined by

$$\frac{1+h(z)}{1-h(z)} = \frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \left\{ \frac{\left(h_\nu^{(2)}(z; q) \right)'}{\left((h_\nu^{(2)})_m(z; q) \right)'} - \delta \right\},$$

where $\delta = \frac{16(1-q)(1-q^\nu)((1-q)(1-q^\nu) - q^\nu) + 2q^{2\nu}}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}}$. The last equality is equivalent to

$$(2.16) \quad \frac{1+h(z)}{1-h(z)} = \frac{1 + \sum_{n=1}^m (n+1) K_n z^n + \frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n=m+1}^{\infty} (n+1) K_n z^n}{1 + \sum_{n=1}^m (n+1) K_n z^n}.$$

By using the equality (2.16) we get

$$h(z) = \frac{\frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n=m+1}^{\infty} (n+1) K_n z^n}{2 + 2 \sum_{n=1}^m (n+1) K_n z^n + \frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n=m+1}^{\infty} (n+1) K_n z^n}$$

and

$$|h(z)| \leq \frac{\frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n=m+1}^{\infty} (n+1) |K_n|}{2 - 2 \sum_{n=1}^m (n+1) |K_n| - \frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n=m+1}^{\infty} (n+1) |K_n|}.$$

The inequality

$$(2.17) \quad \sum_{n=1}^m (n+1) |K_n| + \frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n=m+1}^{\infty} (n+1) |K_n| \leq 1$$

implies that $|h(z)| \leq 1$. It suffices to show that the left hand side of (2.17) is bounded above by

$$\frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n \geq 1} (n+1) |K_n|,$$

which is equivalent to

$$\delta \sum_{n \geq 1} (n+1) |K_n| \geq 0.$$

Thus, the result (2.12) is proved.

To prove the result (2.13), consider the function $k(z)$ defined by

$$\frac{1+k(z)}{1-k(z)} = \left\{ 1 + \frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \right\} \left\{ \frac{(h_\nu^{(2)}(z; q))'}{(h_\nu^{(2)})_m(z; q)}' - \frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \right\}.$$

The last equality is equivalent to

$$(2.18) \quad \frac{1+k(z)}{1-k(z)} = \frac{1 + \sum_{n=1}^m (n+1) K_n z^n - \frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n=m+1}^{\infty} (n+1) K_n z^n}{1 + \sum_{n \geq 1} (n+1) K_n z^n}.$$

From the equality (2.17) we have

$$k(z) = \frac{-\frac{16(1-q)^2(1-q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n=m+1}^{\infty} (n+1) K_n z^n}{2 + 2 \sum_{n=1}^m (n+1) K_n z^n - \delta \sum_{n=m+1}^{\infty} (n+1) K_n z^n}$$

and

$$|k(z)| \leq \frac{\frac{16(1-q)^2(1-q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n=m+1}^{\infty} (n+1) |K_n|}{2 - 2 \sum_{n=1}^m (n+1) |K_n| - \delta \sum_{n=m+1}^{\infty} (n+1) |K_n|}.$$

The inequality

$$(2.19) \quad \sum_{n=1}^m (n+1) |K_n| + \frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n=m+1}^{\infty} (n+1) |K_n| \leq 1$$

implies that $|k(z)| \leq 1$. Since the left hand side of (2.19) is bounded above by

$$\frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n \geq 1} (n+1) |K_n|,$$

which is equivalent to

$$\delta \sum_{n=m+1}^{\infty} (n+1) |K_n| \geq 0,$$

the proof of result (2.13) is completed. \square

Theorem 3. Let $\nu > -1, q \in (0, 1)$, the function $h_\nu^{(3)} : \mathcal{U} \rightarrow \mathbb{C}$ be defined by (1.5) and its sequences of partial sums by $(h_\nu^{(3)})_m(z; q) = z + \sum_{n=1}^m T_n z^{n+1}$. If the inequality $(1-q)(1-q^\nu) \geq 2\sqrt{q}$ is valid, then the next two inequalities are valid for $z \in \mathcal{U}$:

$$(2.20) \quad \Re \left\{ \frac{h_\nu^{(3)}(z; q)}{(h_\nu^{(3)})_m(z; q)} \right\} \geq \frac{(1-q)(1-q^\nu) - 2\sqrt{q}}{\sqrt{q}},$$

$$(2.21) \quad \Re \left\{ \frac{(h_\nu^{(3)})_m(z; q)}{h_\nu^{(3)}(z; q)} \right\} \geq \frac{(1-q)(1-q^\nu) - \sqrt{q}}{\sqrt{q}}.$$

Proof. From the inequality (2.3) we have that

$$(2.22) \quad 1 + \sum_{n \geq 1} |T_n| \leq \frac{(1-q)(1-q^\nu)}{(1-q)(1-q^\nu) - \sqrt{q}}.$$

The inequality (2.22) is equivalent to

$$(2.23) \quad \frac{(1-q)(1-q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n \geq 1} |T_n| \leq 1.$$

In order to prove the inequality (2.20), we consider the function $\phi(z)$ defined by

$$\frac{1 + \phi(z)}{1 - \phi(z)} = \frac{(1-q)(1-q^\nu) - \sqrt{q}}{\sqrt{q}} \left\{ \frac{h_\nu^{(3)}(z; q)}{(h_\nu^{(3)})_m(z; q)} - \frac{(1-q)(1-q^\nu) - 2\sqrt{q}}{\sqrt{q}} \right\},$$

which is equivalent to

$$(2.24) \quad \frac{1 + \phi(z)}{1 - \phi(z)} = \frac{1 + \sum_{n=1}^m T_n z^n + \frac{(1-q)(1-q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n=m+1}^{\infty} T_n z^n}{1 + \sum_{n=1}^m T_n z^n}.$$

From the equality (2.24) we obtain

$$\phi(z) = \frac{\frac{(1-q)(1-q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n=m+1}^{\infty} T_n z^n}{2 + 2 \sum_{n=1}^m T_n z^n + \frac{(1-q)(1-q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n=m+1}^{\infty} T_n z^n}$$

and

$$|\phi(z)| \leq \frac{\frac{(1-q)(1-q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n=m+1}^{\infty} |T_n|}{2 - 2 \sum_{n=1}^m |T_n| - \frac{(1-q)(1-q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n=m+1}^{\infty} |T_n|}.$$

The inequality

$$(2.25) \quad \sum_{n=1}^m |T_n| + \frac{(1-q)(1-q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n=m+1}^{\infty} |T_n| \leq 1$$

implies that $|\phi(z)| \leq 1$. It suffices to show that the left hand side of (2.25) is bounded above by

$$\frac{(1-q)(1-q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n \geq 1} |T_n|,$$

which is equivalent to

$$\frac{(1-q)(1-q^\nu) - 2\sqrt{q}}{\sqrt{q}} \sum_{n=1}^m |T_n| \geq 0.$$

The last inequality holds true for $(1-q)(1-q^\nu) \geq 2\sqrt{q}$.

In order to prove the result (2.21), we consider the function $\varphi(z)$ given by

$$\frac{1 + \varphi(z)}{1 - \varphi(z)} = \left(1 + \frac{(1-q)(1-q^\nu) - \sqrt{q}}{\sqrt{q}} \right) \left\{ \frac{(h_\nu^{(3)})_m(z; q)}{h_\nu^{(3)}(z; q)} - \frac{(1-q)(1-q^\nu) - \sqrt{q}}{\sqrt{q}} \right\}.$$

Then from the last equality we get

$$\varphi(z) = \frac{-\frac{(1-q)(1-q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n=m+1}^{\infty} T_n z^n}{2 + 2 \sum_{n=1}^m T_n z^n - \frac{(1-q)(1-q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n=m+1}^{\infty} T_n z^n}$$

and

$$|\varphi(z)| \leq \frac{\frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}} \sum_{n=m+1}^{\infty} |T_n|}{2 - 2 \sum_{n=1}^m |T_n| - \frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}} \sum_{n=m+1}^{\infty} |T_n|}.$$

The inequality

$$(2.26) \quad \sum_{n=1}^m |T_n| + \frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}} \sum_{n=m+1}^{\infty} |T_n| \leq 1$$

implies that $|\varphi(z)| \leq 1$. Since the left hand side of (2.26) is bounded above by

$$\frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}} \sum_{n \geq 1} |T_n|,$$

which is equivalent to

$$\frac{(1-q)(1-q^\nu)-2\sqrt{q}}{\sqrt{q}} \sum_{n=1}^m |T_n| \geq 0.$$

This completes the proof of the theorem. \square

Theorem 4. Let $\nu > -1, q \in (0, 1)$, the function $h_\nu^{(3)} : \mathcal{U} \rightarrow \mathbb{C}$ be defined by (1.5) and its sequences of partial sums by $(h_\nu^{(3)})_m(z; q) = z + \sum_{n=1}^m T_n z^{n+1}$. If the inequality $(1-q)(1-q^\nu) \geq 4\sqrt{q}$, then the next two inequalities are valid for $z \in \mathcal{U}$:

$$(2.27) \quad \Re \left\{ \frac{\left(h_\nu^{(3)}(z; q) \right)'}{\left((h_\nu^{(3)})_m(z; q) \right)'} \right\} \geq \frac{(1-q)^2(1-q^\nu)^2 - 4(1-q)(1-q^\nu)\sqrt{q} + 2q}{2(1-q)(1-q^\nu)\sqrt{q} - q},$$

$$(2.28) \quad \Re \left\{ \frac{\left((h_\nu^{(3)})_m(z; q) \right)'}{\left(h_\nu^{(3)}(z; q) \right)'} \right\} \geq \frac{((1-q)(1-q^\nu) - \sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q} - q}.$$

Proof. From the inequality (2.4) we have that

$$(2.29) \quad 1 + \sum_{n \geq 1} (n+1) |T_n| \leq \left(\frac{(1-q)(1-q^\nu)}{(1-q)(1-q^\nu) - \sqrt{q}} \right)^2.$$

The inequality (2.29) is equivalent to

$$(2.30) \quad \frac{((1-q)(1-q^\nu) - \sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q} - q} \sum_{n \geq 1} (n+1) |T_n| \leq 1.$$

In order to prove the inequality (2.27), we consider the function $\psi(z)$ defined by

$$\frac{1 + \psi(z)}{1 - \psi(z)} = \frac{((1-q)(1-q^\nu) - \sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q} - q} \left\{ \frac{\left(h_\nu^{(3)}(z; q) \right)'}{\left((h_\nu^{(3)})_m(z; q) \right)'} - \lambda \right\},$$

where $\lambda = \frac{(1-q)^2(1-q^\nu)^2 - 4(1-q)(1-q^\nu)\sqrt{q} + 2q}{2(1-q)(1-q^\nu)\sqrt{q} - q}$. The last equality is equivalent to

$$(2.31) \quad \frac{1 + \psi(z)}{1 - \psi(z)} = \frac{1 + \sum_{n=1}^m (n+1) T_n z^n + \frac{((1-q)(1-q^\nu) - \sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q} - q} \sum_{n=m+1}^{\infty} (n+1) T_n z^n}{1 + \sum_{n=1}^m (n+1) T_n z^n}.$$

By using the equality (2.31) we get

$$\psi(z) = \frac{\frac{((1-q)(1-q^\nu)-\sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q}-q} \sum_{n=m+1}^{\infty} (n+1) T_n z^n}{2 + 2 \sum_{n=1}^m (n+1) T_n z^n + \frac{((1-q)(1-q^\nu)-\sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q}-q} \sum_{n=m+1}^{\infty} (n+1) T_n z^n}$$

and

$$|\psi(z)| \leq \frac{\frac{((1-q)(1-q^\nu)-\sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q}-q} \sum_{n=m+1}^{\infty} (n+1) |T_n|}{2 - 2 \sum_{n=1}^m (n+1) |T_n| - \frac{((1-q)(1-q^\nu)-\sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q}-q} \sum_{n=m+1}^{\infty} (n+1) |T_n|}.$$

The inequality

$$(2.32) \quad \sum_{n=1}^m (n+1) |T_n| + \frac{((1-q)(1-q^\nu)-\sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q}-q} \sum_{n=m+1}^{\infty} (n+1) |T_n| \leq 1$$

implies that $|\psi(z)| \leq 1$. It suffices to show that the left hand side of (2.32) is bounded above by

$$\frac{((1-q)(1-q^\nu)-\sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q}-q} \sum_{n \geq 1} (n+1) |T_n|,$$

which is equivalent to

$$\lambda \sum_{n=1}^m (n+1) |T_n| \geq 0.$$

Thus, the result (2.27) is proved.

To prove the result (2.28), consider the function $\rho(z)$ defined by

$$\frac{1+\rho(z)}{1-\rho(z)} = \left\{ 1 + \frac{((1-q)(1-q^\nu)-\sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q}-q} \right\} \left\{ \frac{(h_\nu^{(3)}(z;q))'}{(h_\nu^{(3)})_m(z;q)} - \frac{((1-q)(1-q^\nu)-\sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q}-q} \right\}.$$

The last equality is equivalent to

$$(2.33) \quad \frac{1+\rho(z)}{1-\rho(z)} = \frac{1 + \sum_{n=1}^m (n+1) T_n z^n - \frac{((1-q)(1-q^\nu)-\sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q}-q} \sum_{n=m+1}^{\infty} (n+1) T_n z^n}{1 + \sum_{n=1}^{\infty} (n+1) T_n z^n}.$$

From the equality (2.33) we get

$$\rho(z) = \frac{-\frac{((1-q)(1-q^\nu)-\sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q}-q} \sum_{n=m+1}^{\infty} (n+1) T_n z^n}{2 + 2 \sum_{n=1}^m (n+1) T_n z^n - \frac{((1-q)(1-q^\nu)-\sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q}-q} \sum_{n=m+1}^{\infty} (n+1) T_n z^n}$$

and

$$|\rho(z)| \leq \frac{\frac{((1-q)(1-q^\nu)-\sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q}-q} \sum_{n=m+1}^{\infty} (n+1) |T_n|}{2 - 2 \sum_{n=1}^m (n+1) |T_n| - \frac{((1-q)(1-q^\nu)-\sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q}-q} \sum_{n=m+1}^{\infty} (n+1) |T_n|}.$$

The inequality

$$(2.34) \quad \sum_{n=1}^m (n+1) |T_n| + \frac{((1-q)(1-q^\nu)-\sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q}-q} \sum_{n=m+1}^{\infty} (n+1) |T_n| \leq 1$$

implies that $|\rho(z)| \leq 1$. Since the left hand side of (2.34) is bounded above by

$$\frac{((1-q)(1-q^\nu) - \sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q} - q} \sum_{n \geq 1} (n+1) |T_n|,$$

which is equivalent to

$$\frac{(1-q)(1-q^\nu) ((1-q)(1-q^\nu) - 4\sqrt{q}) + 2q}{2(1-q)(1-q^\nu)\sqrt{q} - q} \sum_{n=1}^m (n+1) |T_n| \geq 0,$$

the proof of result (2.28) is completed. \square

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